Factorising and Solving Polynomials

A *polynomial* is any function of the form $f(x) = c_0 + c_1 x + c_2 x^2 + ... + c_{n-1} n^{n-1} + c_n x^n$ where $c_n \in R$. For example:

$$f(x) = 2x - x^3 \qquad g(x) = 1 + x^2 + 3x^4 \qquad h(x) = 2x^3 - x^2 + 4x - 1$$

A *zero* / *root* of a polynomial *f* is a value *c* such that $f(c) = 0$. For example:

- If $f(x) = 2x + x^2 - x^3$ then 2 is a root since $f(2) = 2 \cdot 2 + 2^2 - 2^3 = 0$.
- If $f(x) = x^4 - 2x + 1$ then 1 is a root since $f(1) = 1^4 - 2 \cdot 1 + 1 = 0$.

If *c* is a root of a polynomial, then $(x - c)$ will always be a factor of the polynomial. Therefore, if we know a root of a polynomial we can use it to help us factorise the polynomial.

Example

Given that 2 is a root of $f(x) = x^3 - x - 6$, factorise $f(x)$.

Solution

Since 2 is a root we must be able to find $g(x)$ such that:

$$x^3 - x - 6 = (x - 2) \cdot g(x)$$

$$\begin{aligned} x^3 - x - 6 &= (x - 2)(x^2 + ...) \\ &= x^3 - 2x^2 + ... \end{aligned}$$

We must have x^2 because when we expand we need x^3.

$$\begin{aligned} x^3 - x - 6 &= (x - 2)(x^2 + 2x + ...) \\ &= x^3 - 2x^2 + 2x^2 - 4x + ... \\ &= x^3 - 4x + ... \end{aligned}$$

In the previous step the coefficient of x^2 in our expansion is -2, but in our original function the coefficient of x^2 is zero, so we need to introduce $2x$ to our factorization. This will cause the x^2 terms to cancel upon expansion.

$$\begin{aligned} x^3 - x - 6 &= (x - 2)(x^2 + 2x + 3) \\ &= x^3 - \cancel{2x^2} + \cancel{2x^2} - 4x + 3x - 6 \\ &= x^3 - x - 6 \end{aligned}$$

In the previous step the coefficient of x in our expansion is -4, but the coefficient of x in the original function is -1, so we need to introduce 3 to our factorization. This will give an overall coefficient of -1. It will also create a constant term of -6, which is what we need.

Sometimes it is possible to factorise further by factorising $g(x)$. In this case it is not possible. Our solution is therefore:

$$x^3 - x - 6 = (x - 2)(x^2 + 2x + 3)$$

Example

Factorise $x^3 - 5x^2 + 2x + 8$.

Solution

We first need to find a root of the equation. Testing a few values of x gives:

When $x = 1$: $1^3 - 5 \cdot 1^2 + 2 \cdot 1 + 8 = 6$

When $x = 2$: $2^3 - 5 \cdot 2^2 + 2 \cdot 2 + 8 = 0$

So, 2 is a root of the function. This means we can write it in the form $x^3 - 5x^2 + 2x + 8 = (x-2) \cdot g(x)$.

Using the same method as in the previous question we get:

$$
\begin{aligned}
x^3 - 5x^2 + 2x + 8 &= (x-2)(x^2 + \ldots) \\
&= x^3 - 2x^2 + \ldots
\end{aligned}
$$

We must have x^2 because when we expand we need x^3.

$$
\begin{aligned}
x^3 - 5x^2 + 2x + 8 &= (x-2)(x^2 - 3x \ldots) \\
&= x^3 - 2x^2 - 3x^2 + 6x \ldots \\
&= x^3 - 5x^2 + 6x
\end{aligned}
$$

In the previous step the coefficient of x^2 in our expansion is -2, but in our original function the coefficient of x^2 is -5, so we need to introduce $-3x$ to our factorization.

$$
\begin{aligned}
x^3 - 5x^2 + 2x + 8 &= (x-2)(x^2 - 3x - 4) \\
&= x^3 - 5x^2 + 6x - 4x + 8 \\
&= x^3 - 5x^2 + 2x + 8
\end{aligned}
$$

In the previous step the coefficient of x in our expansion is 6, but the coefficient of x in the original function is 2, so we need to introduce -4 to our factorization. This will also create a constant term of 8, which is what we need.

In this example we can factorise further:

$$
\begin{aligned}
x^3 - 5x^2 + 2x + 8 &= (x-2)(x^2 - 3x - 4) \\
&= (x-2)(x-4)(x+1)
\end{aligned}
$$

With practice you should be able to do all of these steps on one line.

Note that whenever you are given an expression to factorise, but are not given a root, you will usually be able to find a root by looking at integer values from – 2 to 2.

Level 1 – 2

1. Use the given root to factorise the following functions:

a) $f(x) = x^3 + 3x^2 + x - 5$ $\qquad f(1) = 0$

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b) $f(x) = x^3 - 3x^2 + 4$ $\qquad f(-1) = 0$

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c) $f(x) = x^3 - 7x - 6$ $\qquad f(3) = 0$

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d) $f(x) = x^3 - x^2 - 22x + 40$ $\qquad f(4) = 0$

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Level 3 – 4

2. Factorise the following functions:

a) $f(x) = x^3 - 19x + 30$

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b) $f(x) = x^3 + 3x^2 - 28x - 60$

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3. Solve the following equations by factorising:

c) $0 = x^3 - 3x^2 - 25x - 21$

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d) $0 = x^3 + 5x^2 - 34x - 80$

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Level 5 – 6

4. Determine the cubic equation that crosses the x-axis at (3,0), (-2,0) and (6,0) and also passes through the point (4,-24).

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5. Determine the cubic equation that crosses the x-axis at (1,0), (2,0) and (4,0) and crosses the y-axis at (0,-6)

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6. Solve the following:

a) $0 = 6x^3 + 3x^2 - 15x + 6$

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b) $0 = x^4 + 2x^3 - 13x^2 - 14x + 24$

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Level 7 – 8

7. If c is a root of $f(x)$ then we can write $f(x)$ as $f(x) = (x - c) \cdot g(x)$ for some function $g(x)$. This is the method that we have been using. But why does it work?

Imagine that there is a remainder $R(x)$ when we divide $f(x)$ by $(x - c)$. For example, when we divide 20 by 7 the remainder is 6:

$$20 = 2 \times 7 + 6$$

Using functions, the idea is the same. For example:

$$f(x) = (x - c) \cdot g(x) + R$$

Explain why, if c is a root of $f(x)$, then R must always be zero.

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8. When $x^3 + 3x^2 + Cx + 1$ is divided by $(x - 2)$ the remainder is 19. Determine the value of C.

Hint: Use the notes in the previous question to help.

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9. When $x^3 + Bx - C$ is divided by $(x+1)$ the remainder is -7. When it is divided by $(x+2)$ the remainder is -16. Determine the values of B and C.

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