Proof of Irrationality

Rational numbers are numbers which can be written as a fraction. For example, the following numbers are rational:

4 3.2 $\frac{1}{2}$ -5.1 -0.25

Written as a fraction they are:

$\frac{4}{1}$ $\frac{16}{5}$ $\frac{1}{2}$ $-\frac{51}{10}$ $-\frac{1}{4}$

Numbers which cannot be written as a fraction are called irrational numbers. An example of an irrational number is π .

Level 1 – 2

1. Let p be a rational number and q be an irrational number. This means that p can be written as $p = \frac{a}{b}$ where a and b are integers.

   Suppose that pq (p multiplied by q) is a rational number. This means that it can be written as $pq = \frac{m}{n}$ where m and n are integers.

   a) Determine an expression for q in terms of a, b, m, and n.

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   b) Hence explain why a rational number multiplied by an irrational number must also be irrational.

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   Note that this method is called proof by contradiction. It involves assuming something is true, and then using that assumption to derive something which contradicts it.

Level 3 – 4

2. Answer the following questions to prove that $\sqrt{2}$ is irrational:

Assume that $\sqrt{2}$ is rational and is written as a fraction $\frac{a}{b}$ where *a* and *b* are in the lowest terms:

$$\sqrt{2} = \frac{a}{b}$$

a) Explain why it is impossible for *both a* and *b* to be even numbers.

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b) Show that a^2 must be even.

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c) Explain why *a* must also be even.

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Since *a* is even, it can be written in the form $a = 2p$ for some integer *p*.

d) Determine an expression for b^2 in terms of *p*.

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e) What can you say about the value of both b^2 and *b*?

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f) Hence, explain why $\sqrt{2}$ must be irrational.

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Level 5 – 6

3. Answer the following questions to prove that $\sqrt{3}$ is irrational:

Assume that $\sqrt{3}$ is rational and is written as a fraction $\frac{a}{b}$ where *a* and *b* are in the lowest terms:

$$\sqrt{3} = \frac{a}{b}$$

a) Determine an expression for a^2 in terms of *b*.

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b) Explain why *b* cannot be even. *Hint: if b is even what can you say about a?*

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c) Hence, explain why both *a* and *b* are odd.

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d) Since *a* and *b* are both odd they can be written as $a = 2m + 1$ and $b = 2n + 1$ for some integers *m* and *n*. Show that $2(m^2 + m) = 6n^2 + 6n + 1$.

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e) Hence, explain why $\sqrt{3}$ must be irrational. *Hint: think about whether each side of the equation in d) is odd or even.*

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Level 7 – 8

4. Using a similar method to the previous question, show that $\sqrt{5}$ is irrational. Explain your reasoning clearly.

5. Show that $\sqrt{6}$ is irrational. Explain your reasoning clearly.