Travel Time

We use various types of transportation in order to travel from one place to another. In Alaska, there is a train called the Hurricane Turn bound between Talkeetna and Hurricane Gulch ("Hurricane Turn Train"). But this train is unique from other trains. It provides flag-stop service, meaning there are no permanent train stops along the tracks ("Hurricane Turn Train"). The train stops anywhere along the tracks if it finds any willing passengers who want to board.

Image taken from Wikimedia Commons

The citizens who live in cabins off-grid just have to walk towards the train tracks, and the train will pick them at any point. However, instead of going straight towards the train tracks, they can travel at a diagonal direction, minimizing the overall travel time.

The aim of this investigation is to find out according to which transportation you take to travel towards the train tracks, what diagonal route you would have to take in order to arrive at your destination in the minimum amount of time.

I will first have to create a scenario by creating a diagram in Geogebra. When creating a scenario I will input my own rules such as distance and speeds of various transportations. With limited information I can label various variables and create a function for time. By using optimization and derivatives I can find the minimum travel time.

Creating a scenario

During the winter, an Alaskan citizen Jack wants to visit his friend, whose cabin is right next to the train tracks (point B). The direct distance from Jacks house (point A) to his friends cabin is 70km. Jack lives directly 30km away from the train tracks.

Jack can get on the train on any point on the train tracks (point D). He will first use a snowmobile which travels at 30km/h to get to the train tracks. The train travels at 60km/h.

Diagram 1 shows the route with all the information we know so far. Type equation here.

The red line shows the route Jack will have to take.

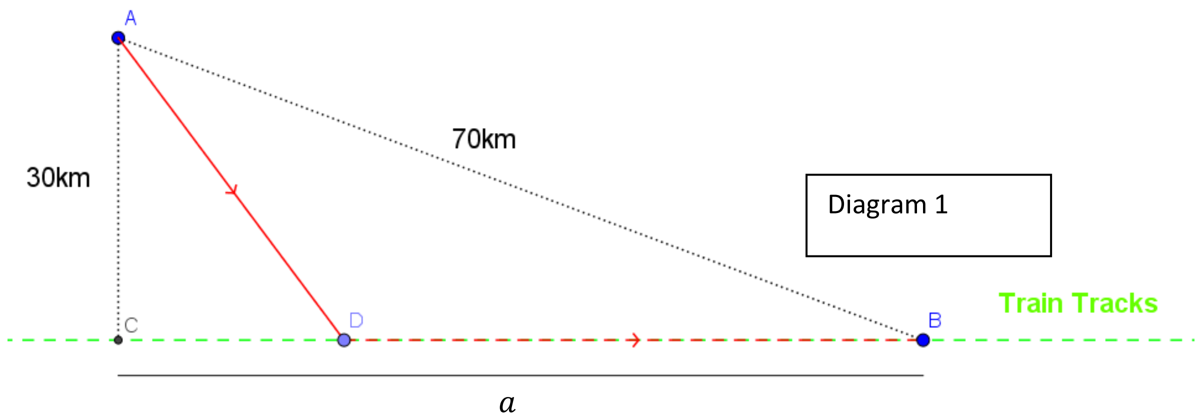


Diagram 1

Step 1: Finding the distances

We will be using the time equation $Time = \frac{Distance}{Speed}$ with the distance representing the length of the red line.

But in order to create distance equations for AD and DB we have to know the value of a. We can apply the Pythagorean Theorem to the triangle ΔABC in order to find the distance of BC or a.

$Pythagorean\ Theorem = a^2 + b^2 = c^2$

$$AC^2 + BC^2 = AB^2$$

$$30^2 + a^2 = 70^2$$

$$a = \sqrt{70^2 - 30^2}$$

$$a = 63.246\ km$$

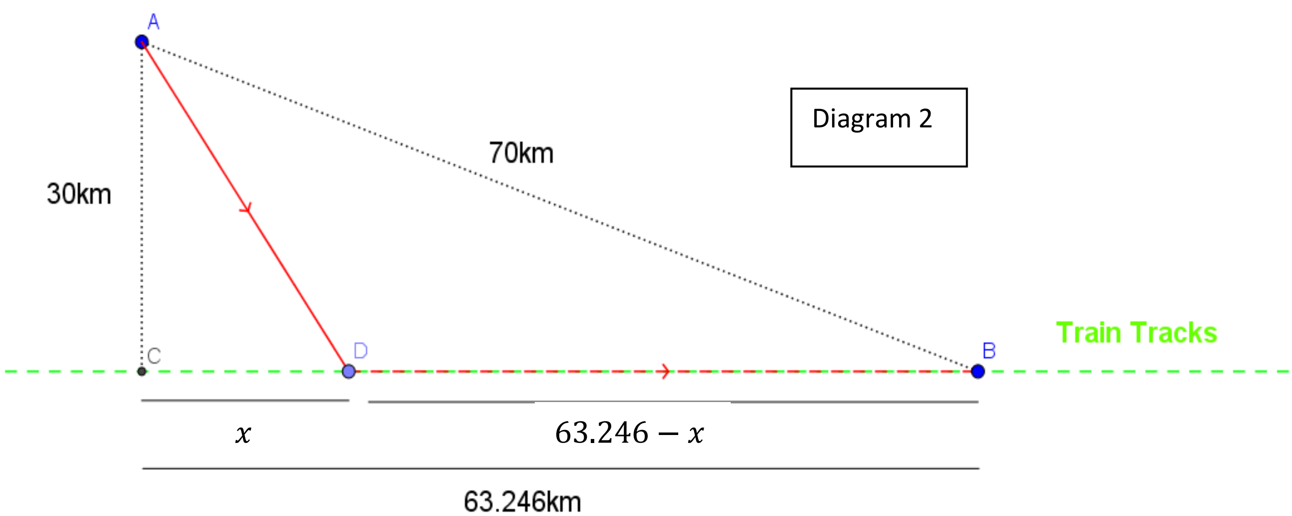
Now we know the total distance from C to B. We can create two distance equations for CD and DB.

Let the distance from point C to where Jack boards the train (point D) be x.

We can therefore say that the distance DB will be

$$63.246 - x$$

Diagram 2 shows all the new information.



We still need to create an equation for distance from A to D. We can again apply the Pythagorean Theorem to ΔACD.

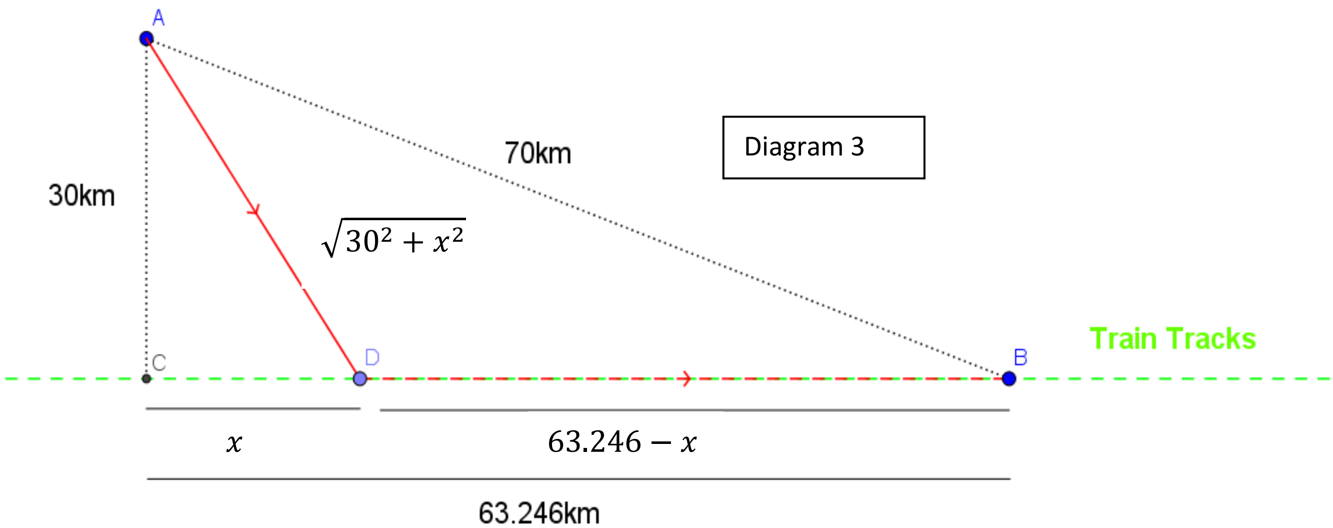
$$AC^2 + CD^2 = AD^2$$

Let distance AD be represented as y:

$$30^2 + x^2 = y^2$$

$$y = \sqrt{30^2 + x^2}$$

We finally have the equations for all the distances of Jack's red route.



Step 2: Creating a function for total travel time (of the red route)

Next we can create an equation for the total time taken to travel.

The snowmobile has a speed of 30km/h and the train travels at 60km/h.

$$Time = \frac{Distance}{Speed}$$

$$Time = \frac{Distance\ travelled\ by\ snowmobile}{Snowmobile\ speed} + \frac{Distance\ travelled\ by\ train}{Train\ spee}$$

We then substitute in the distance equations from Diagram 3 and transportation speeds.

$$Time = t(x) = \frac{\sqrt{30^2 + x^2}}{30} + \frac{63.246 - x}{60}$$

Step 3: Differentiate the function

This becomes our time function. In order to find the minimum amount of time to travel from A to B, we need to find the derivative of this function. The derivative shows us the slope of the original function; therefore by using the derivative we can find the maximum and minimum x-values. In order to make calculations easier we can separate the time function into two and combine them at the end.

$$f(x) = \frac{\sqrt{30^2 + x^2}}{30} \qquad g(x) = \frac{63.246 - x}{60}$$

Let us differentiate the first function (on the left).

We can rewrite the function for easier differentiation.

$$f(x) = \frac{\sqrt{30^2 + x^2}}{30}$$

$$f(x) = \frac{1}{30} \times (30^2 + x^2)^{\frac{1}{2}}$$

We need to use the Chain Rule in order to find the derivative:

$$If\ F(x) = (f \text{ o } g)(x)\ then\ the\ derivative\ of\ F(x) is\ F'(x) = f'\big(g(x)\big)g'(x)$$

We can also rewrite this rule as:

$$y = f(u) and\ u = g(x) then\ the\ derivative\ is{:}\ \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\text{Let } u = 30^2 + x^2$$

$$\frac{du}{dx} = 2x \qquad \frac{dy}{dx} = u^{\frac{1}{2}} = \frac{1}{2}u^{-\frac{1}{2}}$$

$$\frac{dy}{du} \times \frac{du}{dx} = \frac{1}{30} \times 2x \times \frac{1}{2}u^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{30} \times xu^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{30} \times \frac{x}{\sqrt{30^2 + x^2}}$$

$$f'(x) = \frac{x}{30\sqrt{30^2 + x^2}}$$

Now we differentiate the second half of the function.

$$g(x) = \frac{63.246 - x}{60}$$

$$g'(x) = \frac{-1}{60}$$

We can finally combine the two parts of the function back into one.

$$t'(x) = \frac{x}{30\sqrt{30^2 + x^2}} - \frac{1}{60}$$

Now we have the derivative, the value of x will give us the minimum and maximum values. In order to find those values, we have to find the values of x when $y = 0$.

$$\frac{x}{30\sqrt{30^2 + x^2}} - \frac{1}{60} = 0$$

$$\frac{x}{30\sqrt{30^2 + x^2}} = \frac{1}{60}$$

$$60x = 30\sqrt{30^2 + x^2}$$

$$2x = \sqrt{30^2 + x^2}$$

$$4x^2 = 30^2 + x^2$$

$$3x^2 = 900$$

$$x = \pm 17.321$$

Since a distance cannot be a negative value we will only be using the positive value of x.

<u>Step 4: Checking the values</u>

In order to check whether this value is the minimum or maximum value we can use the second derivative test. The second derivative will tell us the concavity of the original function at certain points. First we need to find the second derivative of the time function. We can rewrite the function by eliminating the constant $-\frac{1}{60}$ and taking out the constant $\frac{1}{30}$ for now.

$$t^{'}(x) = \frac{x}{30\sqrt{30^2 + x^2}} - \frac{1}{60}$$

$$\frac{1}{30} \quad \frac{dy}{dx} = \frac{x}{\sqrt{30^2 + x^2}}$$

Since we are differentiating a fraction, we need to use the Quotient Rule:

If $\quad f(x) = \frac{g(x)}{h(x)} \quad$ Then the derivative would be $\quad f'(x) = \frac{g'(x)h(x) - h'(x)g(x)}{(h(x))^2}$

$$\frac{1}{30} \quad \frac{dy}{dx} = \frac{1 \times (30^2 + x^2)^{\frac{1}{2}} - x \times \frac{x}{(30^2 + x^2)^{\frac{1}{2}}}}{(30^2 + x^2)}$$

$$\frac{dy}{dx} = \frac{(30^2 + x^2)^{\frac{1}{2}} - \frac{x^2}{(30^2 + x^2)^{\frac{1}{2}}}}{30(30^2 + x^2)}$$

We could then divide the equation into two fractions with the same denominator.

$$\frac{dy}{dx} = \frac{(30^2 + x^2)^{\frac{1}{2}}}{30(30^2 + x^2)} - \frac{\frac{x^2}{(30^2 + x^2)^{\frac{1}{2}}}}{30(30^2 + x^2)}$$

$$\frac{dy}{dx} = \frac{(30^2 + x^2)^{\frac{1}{2}}}{30(30^2 + x^2)} - \frac{x^2}{30(30^2 + x^2)^{\frac{3}{2}}}$$

Now that the second fraction's denominator changed, we will have to make the first fraction have the same denominator as well.

$$\frac{dy}{dx} = \frac{(30^2 + x^2)}{30(30^2 + x^2)^{\frac{3}{2}}} - \frac{x^2}{30(30^2 + x^2)^{\frac{3}{2}}}$$

$$\frac{dy}{dx} = \frac{(30^2 + x^2) - x^2}{30(30^2 + x^2)^{\frac{3}{2}}}$$

$$t''(x) = \frac{30}{(30^2 + x^2)^{\frac{3}{2}}}$$

We then apply the x value we found and input it into the second derivative equation.

$$t''(x) = \frac{30}{(30^2 + 17.321^2)^{\frac{3}{2}}}$$

$$t''(x) = \frac{30}{(30^2 + 17.321^2)^{\frac{3}{2}}}$$

$$= 0.000722$$

We can say that:

$$t''(17.321) > 0 \quad \text{at } x = 17.321$$

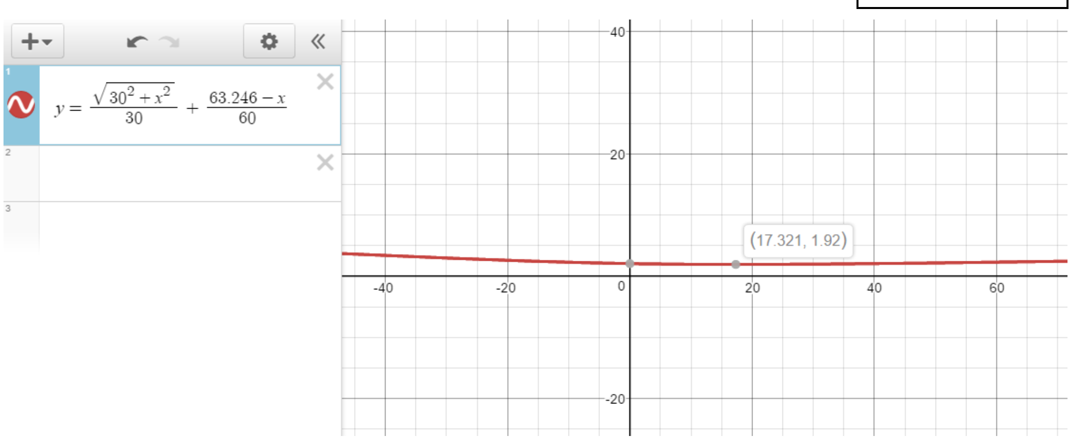
This means that at $x = 17.321$, the graph is concave up, meaning $x = 17.321$ is the minimum local point.

We can check if our calculations are correct by graphing the original function an online graphing calculator (*Desmos Graphing Calculator*). It automatically gives us the minimum value of the function shown in Diagram 4.

$$x = \sqrt{300} = 17.321$$

Therefore our x value is correct.

Diagram 4



Step 5: Find the travel time

We can finally figure out the minimum amount of time by applying the minimum x value into our time equation we made back in the beginning of Step 1.

$$Time = \frac{\sqrt{30^2 + 17.321^2}}{30} + \frac{63.246 - 17.321}{60}$$

$$Time = \frac{\sqrt{1200.017}}{30} + \frac{45.925}{60}$$

$$Time = \frac{\sqrt{1200.017}}{30} + \frac{45.925}{60}$$

$$Time = 1.92\ hours = 1\ hour\ 55.2\ minutes$$

We can conclude that if Jack took the snowmobile→train, through the red route on Diagram 1, it would take him a minimum of 1 hour and 55.2 minutes.

Creating a second scenario

The only difference in the second scenario is that Jack will be using a jeep which travels at 40km/h instead of a snowmobile.

$$Time = \frac{Distance\ travelled\ by\ jeep}{jeep\ speed} + \frac{Distance\ travelled\ by\ train}{Train\ speed}$$

$$Time = \frac{\sqrt{30^2 + x^2}}{40} + \frac{63.246 - x}{60}$$

We have to repeat the same process of finding the derivative and the value of x when $y = 0$.

$$Time'(x) = \frac{x}{40\sqrt{30^2 + x^2}} + \frac{-1}{60}$$

$$\frac{x}{40\sqrt{30^2 + x^2}} + \frac{-1}{60} = 0$$

$$1.5x = \sqrt{30^2 + x^2}$$

$$2.25x^2 = 30^2 + x^2$$

$$x = 26.833$$

We use the original time function and apply the value of x.

$$Time = \frac{\sqrt{30^2 + 26.833^2}}{40} + \frac{63.246 - 26.833}{60}$$

$$Time = 1.613\ hours = 1\ hour\ 36.78\ minutes$$

Creating a third scenario

In the third scenario Jack will be using a dog sleigh which can travel at 15km/h.

Now Jack travels dog sleigh→train from point A to D (Diagram 1)

We can repeat the same process as previous scenarios to find the minimum amount of time taken to travel.

$$Time = \frac{Distance\ travelled\ by\ sleigh}{sleigh\ speed} + \frac{Distance\ travelled\ by\ train}{Train\ speed}$$

$$Time = \frac{\sqrt{30^2 + x^2}}{15} + \frac{63.246 - x}{60}$$

$$Time'(x) = \frac{x}{15\sqrt{30^2 + x^2}} + \frac{-1}{60}$$

$$\frac{x}{15\sqrt{30^2 + x^2}} + \frac{-1}{60} = 0$$

$$60x = 15\sqrt{30^2 + x^2}$$

$$16x^2 = 30^2 + x^2$$

$$x = 7.746$$

We then put this value of x back into the original time function.

$$Time = \frac{\sqrt{30^2 + 7.746^2}}{15} + \frac{63.246 - 7.746}{60}$$

$$Time = 2.991\ hours = 2\ hours\ 59.46\ minutes$$

<u>The relationship between the x value and the speed of transportation</u>

Through this investigation we investigated 3 scenarios and the minimum amount of travel time.

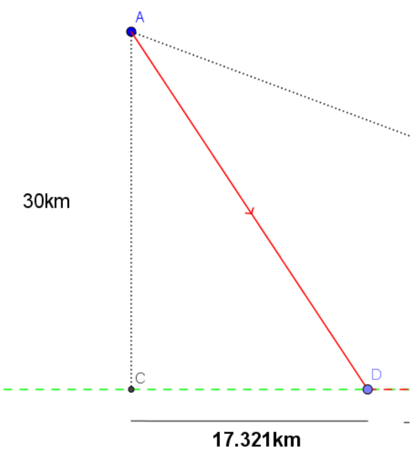
Scenario 1: Snowmobile→Train= 1 hour 55.2 minutes

Scenario 2: Jeep→Train=1 hour 36.78 minutes

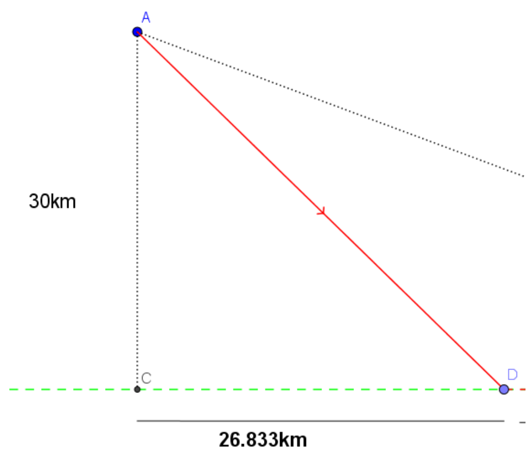
Scenario 3: Sleigh→Train=2 hours 59.46 minutes

The three diagrams below show the x value and the route from A to D for each type of transportation.

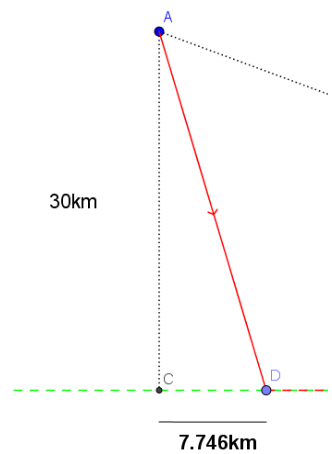
Snowmobile(30km/h)



Jeep(40km/h)



Sleigh(15km/h)



We can see a correlation between the speed of transportation and the x value. The slower the transportation is, the smaller the x value, therefore the shorter the route of AD is in order to achieve the minimum amount of time needed to travel.

Reflection

The limitations to this investigation are that my scenarios are heavily based on assumptions. The Hurricane Train is one of the last flag-stop service trains existing, so the assumption that the train has flag stop service applied to the real world is very low. We also assume that when Jack travels diagonally towards the train tracks he is able to travel in a perfectly straight line with no obstacles in the way. There was also the assumption that all transportations would constantly travel at their given speed even though in real life they may be accelerating or decelerating.

However the methods of using optimization to figure out distance and time is useful for other situations as well because it tells us which route will be best to minimize the travel time. Optimization allows us to figure out the best solution. For example, engineers need to use optimization in order to find the best design. Optimization could also be used in economics, to find the most effective costs under restrictions.

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